Core 3 - June 2006

(1) a) $f(x) = x^3 - x - 7$

$f(2) = 2^3 - 2 - 7 = -1$

$f(2.1) = 2.1^3 - 2.1 - 7 = 0.161$

sign change, so root lies between -1 and 0.161

b) $x^3 - x - 7 = 0$

$x^3 = x + 7$

$x = \sqrt[3]{x + 7}$

c) $x_1 = 2$

$x_2 = \sqrt[3]{2 + 7} = 2.0800\ldots = 2.08$

$x_3 = \sqrt[3]{2.080\ldots + 7} = 2.0862\ldots = 2.09$

$x_4 = \sqrt[3]{2.086\ldots + 7} = 2.0867\ldots = 2.09$

(2) a) $y = (3x - 1)^{10}$ — quick chain rule

$dy/dx = 10 \times 3\,(3x-1)^9 = 30(3x-1)^9$

b) $\int x(2x+1)^8\,dx$ $\qquad u = 2x + 1 \rightarrow x = 1/2(u-1)$

$du/dx = 2 \rightarrow dx = 1/2\,du$

$= \int x\,u^8\,1/2\,du$

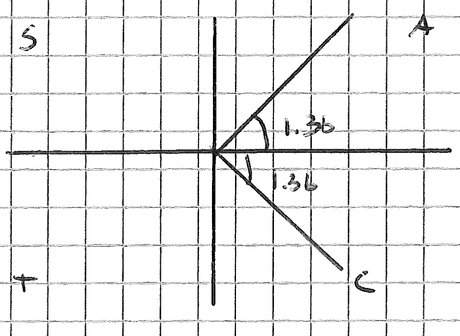
$= \int 1/2(u-1)u^8\,1/2\,du$

$= 1/4 \int u^9 - u^8\,du = 1/4\left[\frac{u^{10}}{10} - \frac{u^9}{9}\right]$

$= \frac{u^{10}}{40} - \frac{u^9}{36} = \frac{(2x+1)^{10}}{40} - \frac{(2x+1)^9}{36} + c$

(3) a) $\sec(x) = 5 \rightarrow 1/\cos(x) = 5 \rightarrow \cos(x) = 0.5$

$x = 1.369\ldots$



$x = 1.37,\ 4.91$

b) $\tan^2(x) = 3\sec(x) + 9$

$\sec^2(x) - 1 = 3\sec(x) + 9$

$\sec^2(x) - 3\sec(x) - 10 = 0$

$\sec^2(x) = 1 + \tan^2(x)$
$\tan^2(x) = \sec^2(x)$

c) $(\sec(x) - 5)(\sec(x) + 2) = 0$

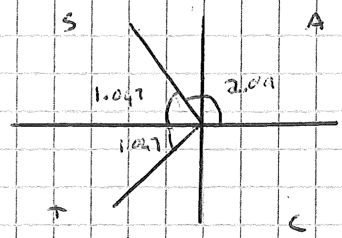
$\sec(x) = 5$

$x = 1.37, 4.91$
(from part a)

$\sec(x) = -2$

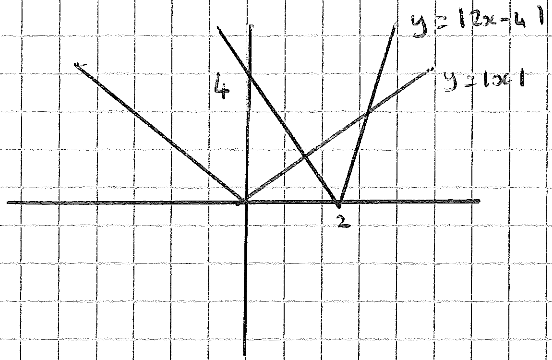
$\cos(x) = -0.5$

$x = 2/3\,\pi = 2.09$
or $x = 4.19$



$x = 1.37, 4.91, 2.09, 4.19$

4) a)



b) i) $|x| = |2x - 4|$

From graph: $x = 2x - 4$ or

$\rightarrow x = 4$

or $-x = 2x - 4$

$4 = 3x$

$x = 4/3$

$\boxed{\text{OR}}$

$x^2 = (2x - 4)^2$

$x^2 = 4x^2 - 16x + 16$

$3x^2 - 16x + 16 = 0$

$(3x - 4)(x - 4) = 0$

$x = 4/3 \quad x = 4$

ii) From graph $4/3 < x < 4$

5) a) $y = e^{2x} - 10e^x + 12x$

i) $dy/dx = 2e^{2x} - 10e^x + 12$

ii) $d^2y/dx^2 = 4e^{2x} - 10e^x$

b) i) $dy/dx = 0 \rightarrow 2e^{2x} - 10e^x + 12 = 0$

$\rightarrow e^{2x} - 5e^x + 6 = 0$

b) ii) Let $z = e^x$

$\rightarrow z^2 - 5z + 6 = 0$

$(z-3)(z-2) = 0$

$z = 3$ $\qquad$ $z = 2$

$e^x = 3$ $\qquad$ $e^x = 2$

$x = \ln(3)$ $\qquad$ $x = \ln(2)$

iii) $y = e^{2x} - 10e^x + 12x$

$\boxed{x = \ln 3}$ $\quad y = e^{2\ln(3)} - 10e^{\ln(3)} + 12(\ln 3)$

$\rightarrow y = 9 - 10 \times 3 + 12\ln(3) = -21 + 12\ln(3)$

$\boxed{x = \ln 2}$ $\quad y = e^{2\ln(2)} - 10e^{\ln(2)} + 12(\ln 2)$

$\rightarrow y = 4 - 10 \times 2 + 12\ln(2) = -16 + 12\ln(2)$

iv) $d^2y/dx^2 = 4e^{2x} - 10e^x$

$\boxed{x = \ln 2}$ $\rightarrow 4e^{2\ln 2} - 10e^{\ln 2}$

$= 4 \times 4 - 10 \times 2 = -4 =$ MAXIMUM

$\boxed{x = \ln 3}$ $\rightarrow 4e^{2\ln 3} - 10e^{\ln 3} =$

$= 4 \times 9 - 10 \times 3 = 6 =$ MINIMUM

6 a)

x	y
1.5	$\ln(1.5)$
2.5	$\ln(2.5)$
3.5	$\ln(3.5)$
4.5	$\ln(4.5)$

$h = 1$

$\rightarrow$ Area $\approx 1 \times (\ln(1.5) + \ln(2.5) + \ln(3.5) + \ln(4.5))$

$= 4.0785... = 4.08$ (3SF)

b) i) $y = x\ln(x)$

$u = x$ $\qquad$ $v = \ln(x)$

$du/dx = 1$ $\qquad$ $dv/dx = 1/x$

PRODUCT RULE

$dy/dx = \ln(x) + x/x$

$= \ln(x) + 1$

ii) EITHER:

$\int \ln(x) + 1 = x\ln(x)$ [FROM b)]

So $\int \ln(x) = x\ln(x) - x + c$

OR:

$\int \ln(x) = \int 1 \times \ln(x)$

$u = \ln(x)$ $\qquad$ $dv/dx = 1$

$du/dx = 1/x$ $\qquad$ $v = x$

USE INTEGRATION BY PARTS!

iii) $\int_1^5 \ln(x)\,dx = \left[x\ln(x) - x\right]_1^5$

$= 5\ln(5) - 5 - 1\ln(1) + 1$

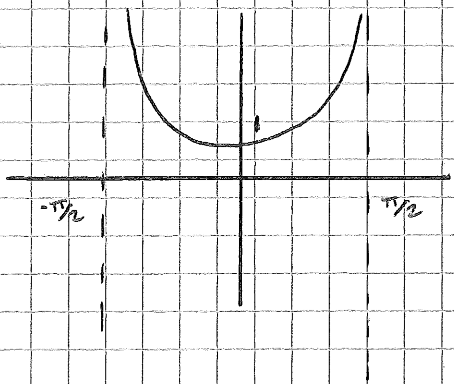
$= 5\ln(5) - 4$

(7) a) $z = \dfrac{\sin(x)}{\cos(x)}$ $\quad u = \sin(x) \quad v = \cos(x)$

$du/dx = \cos(x) \quad dv/dx = -\sin(x)$

$dz/dx = \dfrac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \dfrac{1}{\cos^2(x)} = \sec^2(x)$

b)



c) $y = \sec(x)$

$y^2 = \sec^2(x)$

$\pi \int_0^1 y^2 \, dx$

$= \pi \int_0^1 \sec^2(x)$

$= \pi \left[\tan(x)\right]_0^1$

$= \pi\tan(1) - \pi\tan(0)$

$= 4.8927... = 4.89$

(8) $f(x) = 2e^{3x} - 1$

a) Range : $f(x) > -1$ $\quad$ [Range of e^x is > 0]

b) $y = 2e^{3x} - 1$

$y + 1 = 2e^{3x}$

$\dfrac{y+1}{2} = e^{3x}$

$\ln\left(\dfrac{y+1}{2}\right) = 3x$

$x = \frac{1}{3}\ln\left(\dfrac{y+1}{2}\right) \rightarrow y = \frac{1}{3}\ln\left(\dfrac{x+1}{2}\right) = f^{-1}(x)$

c) $f^{-1}(x) = \frac{1}{3}\ln\left(\dfrac{x+1}{2}\right)$

Let $y = \frac{1}{3}\ln(t)$ $\quad t = \dfrac{x+1}{2}$

$dy/dt = \frac{1}{3t} \quad dt/dx = \frac{1}{2}$

$\therefore dy/dx = \frac{1}{3t} \times \frac{1}{2} = \frac{1}{6t} = \frac{1}{6}\left(\dfrac{2}{x+1}\right)$

when $x = 0$, $dy/dx = 1/6\ (2/1) = 1/3$

9 a) $y = \sin^{-1}(2x)$

when $x = 1/2$, $y = \sin^{-1}(1) = \pi/2$

b) i) $y = \sin^{-1}(2x)$

$\sin(y) = 2x \rightarrow x = 1/2 \sin(y)$

ii) $x = 1/2 \sin(y)$

$dx/dy = 1/2 \cos(y)$

c) If $dx/dy = 1/2 \cos(y)$, $dy/dx = 2/\cos(y)$

From b): $\sin(y) = 2x$

$\sin^2 y + \cos^2 y = 1$

$4x^2 + \cos^2 y = 1$

$\rightarrow \cos^2 y = 1 - 4x^2$

$\cos y = \sqrt{1 - 4x^2}$

$\therefore dy/dx = \dfrac{2}{\sqrt{1 - 4x^2}}$